
Electromagnetic Massless Scalar Field for Cylindrically Symmetric Space-Time

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Abstract :

The paper presents a method to solve Einstein-Maxwell- Massless scalar field equations for cylindrically symmetric universe. Here we have shown that starting from any solution to the electrovacuum field equations, it is possible to generate a whole class of solutions to the electromagnetic massless scalar field equations by suitably redefining one of the metric coefficients. Applying this method to a known solution of the Einstein-Maxwell equations, we have obtained a solution of Einstein-Maxwell-Scalar field equations. Its dual solution is obtained by using an extension of Bonner's theorem [2] to linearly superposed coupled fields.

Key Words : Cylindrical symmetry, electromagnetic field, scalar field, dual solution, coupled fields.

1. INTRODUCTION :

Many researchers have shown their interest towards the study of the relativistic field equations in the presence of a scalar meson field. Brahmachary [3] considered the problem of the coupled gravitational and zero-restmass meson (zero spin) in the case of static spherically symmetric fields. He has shown that no exact solution of the scalar meson field can be found in strictly empty space. However, an approximate solution has been obtained by him which is valid within a certain region. Buchdahl [4] has constructed reciprocal static solutions for axial and spherical fields. This problem has been further discussed by Janis et al. [7] with view of singularities. Their analysis shows that with the addition of zero-rest-mass scalar field, the structure of the event horizon corresponding to $g^{44} = 0$ and $t = \text{constant}$ changes from a non-singular hyper surface to a singular point. They have also Brans-Dicke [1] field equations for static space times and also gave a procedure to generate static solutions of the coupled Einstein-Maxwell scalar. The case of non spherical weyl fields have been studied by Gautreau [6] where as plane symmetric fields has been discussed by Singh [14]. The investigations mentioned above deal with interaction of the scalar meson field (with zero – rest – mass) and the gravitational field. The problem of the scalar meson field of non-zero-rest mass coupled with electromagnetic fields has been examined by Stephenson [19] for static gravitational fields with spherical symmetry.

Roy and Prakesh [10], taking the cylindrically symmetrical metric of Marder [8], have found anisotropic magnetohydrodynamic (M.H.D) cosmological model. Further Singh and Yadav [16] have derived a non-static cosmological model with cylindrical symmetry they have taken the energy momentum tensor to be that of a perfect fluid with an electromagnetic field. Further study in this field has been done by Singh et.al.[15] and Roy and Tripathi [11] by considering by Einstein Maxwell fields. Sharma and Yadav [13] have investigated the problem of coupled gravitational, electromagnetic and scalar fields. It is

found that the energy momentum tensor of massive scalar field can not be the source term for a cylindrically symmetric gravitational field with two degrees of freedom. A similar result in the case of cylindrically symmetric Einstein-Rosen metric with one degree of freedom has been obtained by Roy and Rao [12]. Some other workers in this line are Chitre et al. [17], Bhar [2(a)], Rahman et al. [20].

In this paper we have provided a method to solve electromagnetic and scalar field for space-time with cylindrical symmetry. Here we have found that starting from any solution to the electrovacuum field equations it is possible to generate a whole class of solutions to the Einstein-Maxwell-massless scalar field equations by a suitable redefinition of one of the metric coefficients. Applying this method to a known solution of the Einstein-Maxwell equations we have obtained a solution of Einstein-Maxwell-scalar field equations. Its dual solution is obtained by the use of an extension of Bonnor's theorem [2] to linearly superposed coupled fields.

2. THE FIELD EQUATIONS AND THEIR SOLUTION

We consider the Stachel metric [18] in the form given by

$$(2.1) \quad ds^2 = e^{2\eta-2\delta}(dt^2 - dr^2) - (X^2 e^{2\delta} + r^2 e^{-2\delta})d\phi^2 - e^{2\delta} dz^2 - 2Xe^{2\delta} d\phi dz$$

where η , X and δ depend on r and t only and r , ϕ , z , t levelled as X^1, X^2, X^3, X^4 coordinates. We get Einstein-Rosen metric [5, 9] with one degree of freedom if we take $X = 0$ we consider that distribution with of an electromagnetic field and a scalar zero rest mass meson field V . Thus we have the field equations as follows

$$(2.2) \quad R_{\alpha\beta} = -k[E_{\alpha\beta} + V, \alpha V, \beta],$$

$$(2.3) \quad E_{\alpha\beta} = -g^{ab}F_{\alpha}{}^a F_{\beta}{}^b + \frac{1}{4}g_{\alpha\beta}F_{ab}F^{ab}$$

$$(2.4) \quad F[\alpha, \beta, k] = 0,$$

$$(2.5) \quad F^{\alpha\beta};\beta = 0,$$

$$(2.6) \quad g^{\alpha\beta}V;\alpha\beta = 0$$

For the metric (2.1) the components R_{12} , R_{13} , R_{24} and R_{34} vanish identically. This leads to

$$(2.7) \quad -F_{14}F_{24}e^{2\delta-2\eta} = F_{12}F_{23}\frac{xe^{-2\delta}}{r^2} - F_{13}F_{23}\frac{e^{2\delta}}{r^2}$$

$$(2.8) \quad E_{12}F_{32}\left[e^{-2\delta} + \frac{x^2 e^{2\delta}}{r^2}\right] = F_{14}F_{34}e^{2\delta-2\eta} + F_{13}F_{32}\frac{x}{r^2}e^{2\delta}$$

$$(2.9) \quad F_{32}F_{42}\left[e^{-2\delta} + \frac{x^2 e^{2\delta}}{r^2}\right] = F_{32}F_{43}\frac{xe^{2\delta}}{r^2} - F_{31}F_{41}e^{2\delta-2\eta},$$

$$(2.10) \quad F_{41}F_{21}e^{2\delta-2\eta} = F_{23}F_{42}\frac{x}{r^2}e^{2\delta} - F_{23}F_{43}\frac{1}{r^2}e^{2\delta}$$

In view of (2.7) – (2.10) two cases may arise

$$(i) \quad F_{23}, F_{14} \neq 0 \text{ and } F_{12}, F_{13}, F_{24}, F_{34} \text{ all vanish}$$

(ii) $F_{12}, F_{13}, F_{24}, F_{34} \neq 0$ and F_{14} and F_{23} vanish

We can discuss field components F_{ij} in terms of two potentials θ_2 and θ_3 and consider only case (ii)

Hence

$$(2.11) \quad F_{ij} = \theta_{i;j} - \theta_{j;i}$$

Thus we have

$$F_{12} = -\phi_1, F_{13} = -\psi_1, F_{24} = \phi_4, F_{34} = \psi_4$$

where

$$\theta_2 = \phi, \theta_3 = \psi \text{ and } \phi_i = \phi, \psi_i = \psi$$

Here also we are taking the same live element (2.1) but here η, X, δ are functions of r and t both. Also equation (2.6) leads to

$$\nabla^2 V \equiv V_{,11} - V_{,44} + \left(V, \frac{1}{r} \right) = 0$$

The field equation (2.2), (2.5) and (2.6) lead to

$$(2.12) \quad \eta_1 = r(\delta_1^2 + \delta_4^2) + \frac{e^{4\delta}}{4r}(X_1^{2\delta} + X_4^2) \\ + \frac{1}{2} \left[r(v_1^{2\delta} + v_4^2) + \frac{e^{2\delta}}{r}(\phi_1^2 + \phi_4^2) \right. \\ \left. + \left(\frac{X^2 e^{2\delta}}{r^2} + r e^{-2\delta} \right) (\psi_1^2 + \psi_4^2) \right. \\ \left. - 2X e^{2\delta} (\phi_1 \psi_1 + \phi_4 \psi_4) \right]$$

$$(2.13) \quad \eta_4 = \frac{X_1 X_4 e^{4\delta}}{2r} + 2r \delta_1 \delta_4 + K \left[r(v_{,1} v_{,4} \right. \\ \left. + \frac{e^{2\delta}}{r}(\phi_1 \phi_4) + \left(\frac{X^2 e^{2\delta}}{r} + r e^{-2\delta} \right) (\psi_1 \psi_4) \right. \\ \left. - \frac{c}{r} e^{2\delta} (\phi_1 \psi_4) - \frac{X}{r} e^{2\delta} (\psi_1 \phi_4) \right]$$

$$(2.14) \quad \left[(\delta_1^2 - \delta_4^2) - \frac{e^4}{4r^2} (X_1^2 - X_4^2) + (\eta_{11} - \eta_{44}) \right] \\ = \frac{1}{2} \left[(v_{,4}^2 - v_{,1}^2) + \frac{e^2}{r^2} (\phi_1^2 - \phi_4^2) - \left(\frac{K^2 e^{2\delta}}{r^2} + e^{-2\delta} \right) \right. \\ \left. (\psi_1^2 - \psi_4^2) + \frac{2X}{2} e^2 (\phi_4 \psi_4 - \phi_1 \psi_1) \right]$$

$$(2.15) \left[(\mathbf{X}_{11} - \mathbf{X}_{44}) + 4(\delta_1 \mathbf{X}_1 - \delta_4 \mathbf{X}_4) - \frac{\mathbf{X}_1}{r} \right] \\ = 2\mathbf{K}e^{2\delta} \left[(\phi_4 \psi_4 - \phi_1 \psi_1) + \mathbf{X}(\psi_1^2 - \psi_4^2) \right]$$

$$(2.16) \left[(\delta_{11} - \delta_{44}) + \frac{e^{4\delta}}{2r^2} (\mathbf{X}_4^2 - \mathbf{X}_1^2) + \frac{\mathbf{X}_1}{r} \right] \\ = \frac{1k}{2} \left[\frac{e^{2\delta}}{r^2} (\phi_1^2 - \phi_4^2) + \left(\frac{\mathbf{X}^2 e^{2\delta}}{r^2} - e^{-2\delta} \right) (\psi_1^2 - \psi_4^2) \right. \\ \left. - \frac{2\mathbf{X}e^{2\delta}}{r^2} (\phi_1 \psi_1 - \phi_4 \psi_4) \right].$$

$$(2.17) \left[\eta_{11} - \eta_{44} + \delta_1^2 - \delta_4^2 - \frac{e^{4\delta}}{4r^2} (\mathbf{X}_1^2 - \mathbf{X}_4^2) \right] \\ = \frac{1k}{2} \left[(v_4^2 - v_1^2) + \frac{e^{2\delta}}{r^2} (\phi_1^2 - \phi_4^2) + \left(\frac{\mathbf{X}^2 e^{2\delta}}{r^2} - e^{-2\delta} \right) \right. \\ \left. (\psi_1^2 - \psi_4^2) + \frac{2\mathbf{X}}{r^2} e^{2\delta} (\phi_4 \psi_4 - \phi_1 \psi_1) \right].$$

$$(2.18) (\mathbf{X}_4 \psi_4 - \mathbf{X}_1 \psi_1) + \mathbf{X}(\psi_{44} - \psi_{11}) - (\phi_{44} - \phi_{11}) \\ + 2\mathbf{X}(\psi_4 \delta_4 - \psi_1 \delta_1) - 2(\phi_4 \delta_4 - \phi_1 \delta_1) \\ + \frac{1}{r} (\mathbf{X} \psi_1 - \phi_1) = 0$$

$$(2.19) [\mathbf{X}^2 (\psi_{11} - \psi_{44}) + 2c(\mathbf{X}_1 \psi_1 - \mathbf{X}_4 \psi_4) \\ - \mathbf{X}(\phi_{11} - \phi_{44}) - (\phi_1 \mathbf{X}_1 - \phi_4 \mathbf{X}_4) \\ + 2\{(\mathbf{X}^2 \psi_1 - \mathbf{X} \phi_1) \delta_1 + (-\mathbf{X}^2 \psi_4 + \mathbf{X} \phi_4) \delta_4\} \\ - \frac{1}{r} (\mathbf{X}^2 \psi_1 - \mathbf{X} \phi_1) + r^2 e^{-4\delta} (\psi_{11} - \psi_{44}) \\ - 2r^2 e^{-4\delta} (\psi_1 \delta_1 - \psi_4 \delta_4) + r e^{-4\delta} \psi_1 = 0.$$

$$(2.20) \mathbf{V}_{11} - \mathbf{V}_{44} + \frac{\mathbf{V}_1}{r} = 0.$$

Equations (2.12), (2.13) and (2.17) enable us to define

$$\eta = \eta^v + \eta^v$$

where η^v , corresponding to EM field equations, can be evaluated by integration once a solution for δ , \mathbf{X} , ϕ and ψ are known and η^v depends only on the scalar field \mathbf{V} as

$$(2.21) (\eta^v)_1 = \frac{1}{2} kr (\mathbf{V}_1^2 + \mathbf{V}_4^2)$$

$$(2.22) \quad (\eta^V)_4 = kr V_1 V_4$$

Which again can be evaluated by integration once a solution of equation (2.20) is specified. Thus we may state the following theorem whose application gives a solution of the EMS field equations for every solution of the EMS equations.

Theorem :

If $\delta, X, \eta^V, \theta_2$ and θ_3 form a solution to the electrovacuum field equations for the metric (2.1), then $\delta, X, \theta_2, \theta_3$ and η , where

$$(2.23) \quad \eta = \eta^V + \eta^V$$

and

$$(2.24) \quad (\eta^V)_1 = \frac{1}{2}kr(V_1^2 + V_4^2)$$

$$(2.25) \quad (\eta^V)_4 = kr V_1 V_4$$

$$(2.26) \quad \nabla^2 V = 0$$

is the corresponding solution to the coupled Einstein-Maxwell massless scalar field equations for the metric (2.1). As an example for application of the technique we consider the solution of EM equation given by (Chitre et. al. [17]).

$$(2.27) \quad ds^2 r^{-4/9} e^{(\lambda^2 r^2)/3} (dt^2 - dr^2) - r^{4/3} dz^2 - r^{2/3} (d\phi + \lambda r^{2/3} dz)^2$$

where λ is constant and

$$(2.28) \quad \begin{cases} \theta_2 = \sqrt{2}ar^{2/3} \\ \theta_3 = \frac{\lambda^2}{\sqrt{2}}r^{4/3} \end{cases}$$

On comparing the metric (2.27) with (2.1) we get

$$(2.29) \quad \begin{cases} \eta = \frac{1}{2} \left[\frac{\lambda^2}{3} r^2 + \text{Log}(r^{4/3} + 4r^2) - \frac{4}{9} \text{Log} r \right] \\ \delta = \frac{1}{2} \text{Log} [r^{4/3} + \lambda^2 r^2] \\ X = \frac{\lambda r^{4/3}}{[r^{4/3} + \lambda^2 r^2]} \end{cases}$$

The corresponding solution for the Einstein-Maxwell-Scalar field equations is given by

$$(2.30) \quad \left[\begin{array}{l} \delta = \frac{1}{2} \log [r^{4/3} + \lambda^2 r^2] \\ X = \frac{\lambda r^{4/3}}{[r^{4/3} + \lambda^2 r^2]} \\ V = \text{Log } r \\ \eta = \frac{1}{2} \left[\frac{\lambda^2}{3} r^2 + \text{Log}(r^{4/3} + \lambda^2 r^2) - \frac{4}{9} \text{Log } r - \frac{\mu^2}{r^2} \right] \\ \theta_2 = \sqrt{2} 2r^{2/3}, \theta_3 = 2\sqrt{2} r^{4/3} \end{array} \right.$$

for the metric (2.1)

The theorem can easily be applied to other known solutions of the EM equation and the solution of EMS equations are obtained without integration.

3. AN APPLICATION OF BONNER'S THEOREM

Bonnor [2] has established a theorem which states that

If (g_{ij}, F_{ij}) is a solution of Einstein-Maxwell field equations, then (g_{ij}, F_{ij}^*) is also a solution for the same field equations where

$$(3.1) \quad \begin{aligned} F^{ij} &= \frac{1}{2} \eta^{ijkl} F_{kl} \\ &= -\frac{1}{2} \varepsilon^{ijkl} (-g)^{-1/2} F_{kl} \end{aligned}$$

is the dual of F_{ij} , ε_{ijkl} is the usual permutation symbol and g is the determinant of the metric tensor g_{ij} . The result of Bonnar is also true for linearly superposed coupled fields. This extended duality theorem of Bonnar may be stated as

If (g_{ij}, F_{ij}, V) is a solution of Einstein-Maxwell-Scalar field equations, then $(g_{ij}, F_{ij,v}^*)$ is also a solution for the same field equations.

Now we consider the metric (2.1) and the solution (2.30). The non-vanishing components of the electromagnetic field tensor corresponding this solution are

$$(3.2) \quad \begin{aligned} F_{13} &= -F_{31} = \frac{-4\lambda^2 r^{1/3}}{3\sqrt{2}} = -\frac{2\sqrt{2}\lambda^2}{3} r^{1/3} \\ F_{12} &= -F_{21} = \frac{-2\lambda\sqrt{2}}{3} r^{-2/3} \end{aligned}$$

Using (3.1), we have

$$(3.3) \quad F^{*24} = -\frac{2\sqrt{2}\lambda^2}{3} r^{-2/9} e^{-\lambda^2 r^{2/3}}$$

and

$$F^{*34} = \frac{4\lambda}{3\sqrt{2}} r^{-8/9} e^{-\lambda^2 r^{2/3}}$$

Hence another solution of EMS equations is given by

$$(3.4a) \delta = \frac{1}{2} \text{Log} [r^{4/3} + \lambda^2 r^2]$$

$$(3.4b) X = \frac{\lambda r^{4/3}}{[r^{4/3} + \lambda^2 r^2]}$$

$$(3.4c) V = \mu \text{Log } r$$

$$(3.4d) \eta = \frac{1}{2} \left[\frac{1}{3} \lambda^2 r^2 + \text{Log}(r^{4/3} + \lambda^2 r^2) - \frac{4}{9} \text{Log } r + \frac{\mu^2}{r^2} \right]$$

$$(3.5) F^{*24} = -\frac{4\lambda^2}{3\sqrt{2}} r^{-2/9} e^{-\lambda^2 r^{2/3}}$$

$$F^{*34} = \frac{2\sqrt{2}\lambda}{3} r^{-8/9} e^{-\lambda^2 r^{2/3}}$$

with metric (2.1)

4. DISCUSSION

1. A dual solution, however, does not represent a different physical reality and as such need not be treated separately from the physical point of view.
2. It is clear that for a cylindrically symmetric electrovacuum space time with two degrees of freedom, the presence of massless scalar field as an additional source for the geometry merely results in a redefinition of one of the metric coefficients provided the line element is expressed in the (2.1). Thus starting from any known cylindrical solution of the coupled Einstein-Maxwell field equations it is possible to generate solutions to the EMS equations by the application of the theorem stated.

5. REFERENCES

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